UNCONDITIONAL MOTIVIC GALOIS GROUPS AND VOEVODSKY'S NILPOTENCE CONJECTURE IN THE NONCOMMUTATIVE WORLD

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Abstract. In this article we further the study of noncommutative pure motives initiated in [13, 14, 15]. We construct unconditional noncommutative motivic Galois groups and relate them to the unconditional motivic Galois groups developed originally by André-Kahn. Then, we introduce the correct noncommutative analogue of Voevodsky's nilpotence conjecture and explore its interaction with the finite dimensionality of noncommutative Chow motives as well as with Voevodsky's original conjecture.

Motivating questions

Via an evolved "$\otimes$-categorification" of the classical Wedderburn-Malcev's theorem, André and Kahn constructed in [5] *unconditional* motivic Galois groups. These groups Gal_H (attached to every classical Weil cohomology H) are well-defined up to an interior automorphism and do not require the assumption of any of Grothendieck's standard conjectures. This lead us naturally to the following

Question: *Do these unconditional motivic Galois groups* Gal_H *admit noncommutative analogues* Gal_H^{NC} *? If so, what is the relation between* Gal_H *and* Gal_H^{NC} *?*

In a foundational work [18], Voevodsky introduced the smash-nilpotence equivalence relation on algebraic cycles and conjectured its agreement with numerical equivalence. Subsequently, O'Sullivan [16] found a beautiful link between this conjecture and the finite dimensionality of Chow motives. This circle of conjectures motivates the following

Question: *Does Voevodsky's nilpotence conjecture admits a noncommutative analogue ? If so, what is its relation with the finite dimensionality of noncommutative Chow motives and with Voevodsky's original conjecture ?*

In this article we provide precise answers to the above questions; see Theorems 1.7 and 4.1. In the process we show that orbit categories are well-behaved with respect to $\otimes$-nilpotence; see Proposition 2.1.

Preliminaries

Throughout the article we will reserve the letter k for the base field and the letter F for the field of coefficients. The pseudo-abelian envelope construction will be denoted by $(-)^\natural$. We will make use of the language of differential graded (=dg) categories; consult [15, §2] or Keller's ICM address [9]. Given a dg category $\mathcal{A}$ we

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will denote by $\mathcal{A}^{\mathsf{op}}$ its opposite dg category. Moreover, we will denote by $\underline{k}$ the dg category with a single object and with k as the dg algebra of endomorphisms (concentrated in degree zero). Recall that a dg category $\mathcal{A}$ is called *smooth* and *proper* in the sense of Kontsevich [11] if it perfect as a bimodule over itself and for each pair of objects (x,y) the complex $\mathcal{A}(x,y)$ of k-vector spaces verifies the inequality $\sum_{i\in\mathbb{Z}}\dim H^i\mathcal{A}(x,y) < \infty$. A large class of examples of smooth and proper dg categories is provided by the dg enhancements $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z)$ of the derived categories $\mathcal{D}_{\mathsf{perf}}(Z)$ of perfect complexes of $\mathcal{O}_Z$-modules over smooth projective k-schemes Z; consult Lunts-Orlov [12].

We will also assume that the reader is familiar with the categories $\mathrm{Chow}(k)_F$ and $\mathrm{Num}(k)_F$ of Chow and numerical motives (see [2, §4]) as well as with the categories $\mathrm{NChow}(k)_F$ and $\mathrm{NNum}(k)_F$ of noncommutative Chow and numerical motives (see [15, §3]). Recall that in analogy with the commutative world, the morphisms in $\mathrm{NChow}(k)_F$ and $\mathrm{NNum}(k)_F$ are called *noncommutative correspondences*.

Finally, we will make reference to the standard conjectures C and D (see [2, §5]) as well as to the noncommutative standard conjectures C_{NC} and D_{NC} (see [15]).

1. Unconditional motivic Galois groups

Throughout this section we will assume that F is a field of characteristic zero and moreover that k is a field extension of F or vice-versa.

As proved in [15, Thm. 7.2], periodic cyclic homology HP gives then rise to a well-defined F-linear $\otimes$-functor

$$\overline{HP_*}:\mathrm{NChow}(k)_F\longrightarrow \mathrm{sVect}(K) \tag{1.1}$$

with values in the category of finite dimensional super K-vector spaces (with $K=F$ when F is a field extension of k and $K=k$ when k is a field extension of F). Following [15, Def. 10.4], let us denote by $\mathrm{NHom}(k)_F$ the category of *noncommutative homological motives*, i.e. the pseudo-abelian envelope of the quotient category $\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*})$ where $\mathrm{Ker}(\overline{HP_*})$ is the kernel of (1.1). Since by construction the category $\mathrm{sVect}(K)$ is idempotent complete the above functor (1.1) descends to a faithful F-linear $\otimes$-functor

$$\overline{HP_*}:\mathrm{NHom}(k)_F\longrightarrow \mathrm{sVect}(K)\,. \tag{1.2}$$

Now, let us denote by $\mathrm{NHom}(k)_F^{\pm}$ the full subcategory of those noncommutative homological motives N whose associated Künneth projectors

$$\pi_N^{\pm}:\overline{HP_*}(N)\twoheadrightarrow \overline{HP_*}^{\pm}(N)\hookrightarrow \overline{HP_*}(N)$$

are *algebraic*, i.e. that can be written as $\pi_N^{\pm}=\overline{HP_*}(\underline{\pi}_N^{\pm})$ with $\underline{\pi}_N^{\pm}$ noncommutative correspondences. By hypothesis F is of characteristic zero and so K is (in both cases) a semi-simple F-algebra. Hence, by applying [5, Prop. 2] to the functor (1.2) we obtain a new rigid symmetric monoidal category $\mathrm{NHom}^{\dagger}(k)_F^{\pm}$ (obtained from $\mathrm{NHom}(k)_F^{\pm}$ by modifying its symmetric isomorphism constraints) and a composed faithful F-linear $\otimes$-functor

$$\mathrm{NHom}^{\dagger}(k)_F^{\pm}\subset \mathrm{NHom}(k)_F\xrightarrow{\overline{HP_*}}\mathrm{sVect}(K)\longrightarrow \mathrm{Vect}(K)\,, \tag{1.3}$$

where the last functor is the forgetful functor. More importantly, $\mathrm{NHom}^{\dagger}(k)_F^{\pm}$ is semi-primary, its $\otimes$-ideals $\mathcal{R}^{\pm}$ and $\mathcal{N}^{\pm}$ agree, the quotient category

$$\mathrm{NNum}^{\dagger}(k)_F^{\pm}:=\mathrm{NHom}^{\dagger}(k)_F^{\pm}/\mathcal{N}^{\pm}\subset \mathrm{NNum}(k)_F$$

is abelian semi-simple, and the canonical projection $\otimes$-functor

$$\mathrm{NHom}^\dagger(k)^{\pm}_F \longrightarrow \mathrm{NNum}^\dagger(k)^{\pm}_F \tag{1.4}$$

is conservative; consult [5, §1]. By [5, Thm. 8 a)] the projection (1.4) admits then a $\otimes$-section and any two such $\otimes$-sections are conjugated by a $\otimes$-isomorphism. By choising a $\otimes$-section s^{NC} we obtain then a composed faithful F-linear $\otimes$-functor

$$\omega_{HP} : \mathrm{NNum}^\dagger(k)^{\pm}_F \stackrel{s^{NC}}{\longrightarrow} \mathrm{NHom}^\dagger(k)^{\pm}_F \stackrel{(1.3)}{\longrightarrow} \mathrm{Vect}(K)\,.$$

Note that when endowed with ω_{HP}, the category $\mathrm{NNum}^\dagger(k)^{\pm}_F$ becomes a Tannakian category; see [15, Appendix A].

Definition 1.5. The *unconditional noncommutative motivic Galois group* Gal^{NC}_{HP} is the group $\underline{\mathrm{Aut}}^{\otimes}(w_{HP})$ of $\otimes$-automorphisms of the above fiber functor ω_{HP}.

Note that a different choice of the $\otimes$-section gives nevertheless rise to an isomorphic group (via an interior automorphism). Moreover, since $\mathrm{NHom}^\dagger(k)^{\pm}_F$ is abelian semi-simple, Gal^{NC}_{HP} is a *pro-reductive* group, *i.e.* its unipotent radical is trivial.

Assuming the noncommutative standard conjectures C_{NC} and D_{NC}, Gal^{NC}_{HP} identifies with the noncommutative motivic Galois group $\mathrm{Gal}(\mathrm{NNum}^\dagger(k)_F)$ introduced[1] by the authors in [15]. This follows from the fact that by C_{NC} we have the equality $\mathrm{NHom}(k)^{\pm}_F = \mathrm{NHom}(k)_F$ (see [15, §9]) and that by D_{NC} we can choose the $\otimes$-section to be precisely the identity functor (see [15, §10]).

Remark 1.6. Assume that k is a field extension of F. Then, the proof of [15, Thm. 1.3] shows us that for every smooth projective k-scheme Z which satisfies the classical sign conjecture, the associated dg category $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z)$ belongs to $\mathrm{NHom}(k)^{\pm}_F$. As proved by Kleiman [10], this is the case for all abelian varieties. Using the stability of $\mathrm{NHom}(k)^{\pm}_F$ under direct factors and tensor products (see [15, Prop. 8.2]) we obtain then a large class of examples of noncommutative motives belonging to $\mathrm{NHom}(k)^{\pm}_F$ (and hence to $\mathrm{NNum}^\dagger(k)^{\pm}_F$).

Theorem 1.7. (i) *When k is a field extension of F we have a well-defined comparison group homomorphism*

$$\mathrm{Gal}^{NC}_{HP} \to \mathrm{Ker}\left(t : \mathrm{Gal}_{dR} \twoheadrightarrow \mathrm{Gal}_{dR}(\mathbb{Q}(1))\right)\,, \tag{1.8}$$

where $\mathrm{Gal}_{dR}(\mathbb{Q}(1))$ denotes the Galois group of the Tannakian subcategory generated by the Tate motive $\mathbb{Q}(1)$.

(ii) *When $k = F$ the comparison homomorphism (1.8) reduces to*

$$\mathrm{Gal}^{NC}_{HP} \twoheadrightarrow \mathrm{Ker}\left(t : \mathrm{Gal}_{dR} \twoheadrightarrow \mathbb{G}_m\right)\,, \tag{1.9}$$

where $\mathbb{G}_m$ is the multiplicative group.

(iii) *Assuming the standard conjectures C and D as well as the noncommutative standard conjectures C_{NC} and D_{NC}, the above comparison homomorphism (1.9) identifies with the comparison homomorphism*

$$\mathrm{Gal}(\mathrm{NNum}^\dagger(k)_k) \twoheadrightarrow \mathrm{Ker}\left(t : \mathrm{Gal}(\mathrm{Num}^\dagger(k)_k) \twoheadrightarrow \mathbb{G}_m\right)$$

constructed originally in [15, Thm. 1.7].

[1] In *loc. cit.* we have restricted ourselves to neutral Tannakian categories and hence to the case where F is a field extension of k. However, the general case is completely similar.

Intuitively speaking, Theorem 1.7 shows us that the $\otimes$-symmetries of the commutative world which can be lifted to the noncommutative world are precisely those that become trivial when restricted to the Tate motive.

Remark 1.10. As explained by Deligne in [7, §1.6], in the non-neutral setting one should replace Galois groups by Galois groupoids. More precisely, associated to every Tannakian category $\mathcal{T}$ (defined over a field F of characteristic zero) and fiber functor $\omega : \mathcal{T} \to \mathrm{Mod}(k)$ with values in the category of modules over a commutative F-algebra, there is a well-defined Galois groupoid $\underline{\mathrm{Aut}}^{\otimes}(\omega)$. This is a groupoid scheme which acts transitively on $\mathrm{Spec}(k)$ and whose source and target maps give rise to a flat morphism $\underline{\mathrm{Aut}}^{\otimes}(\omega) \to \mathrm{Spec}(k) \times_F \mathrm{Spec}(k)$. In the particular case of the Tannakian subcategory generated by the Tate motive $\mathbb{Q}(1)$ – Theorem 1.7(i) – the associated Galois groupoid is a $\mathbb{G}_m$-bundle over $\mathrm{Spec}(k) \times_F \mathrm{Spec}(k)$ and the images $\omega(\mathbb{Q}(n))$ of the Tate motives become identified with vector bundles over $\mathrm{Spec}(k)$.

Proof. Let us start by proving item (i). Recall from the proof of [15, Thm. 1.7] the construction of the following commutative diagram

$$\begin{tikzcd}
\mathrm{Chow}(k)_F \arrow[r] \arrow[d] & \mathrm{Chow}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r] \arrow[d] & \mathrm{NChow}(k)_F \arrow[r, "\overline{HP_*}"] \arrow[d] & \mathrm{sVect}(k) \arrow[d, equal] \\
\mathrm{Chow}(k)_F/\mathrm{Ker} \arrow[r] \arrow[d] & (\mathrm{Chow}(k)_{F/_{-\otimes\mathbb{Q}(1)}})/\mathrm{Ker} \arrow[r] \arrow[d] & \mathrm{NHom}(k)_F \arrow[r, "\overline{HP_*}"'] \arrow[d] & \mathrm{sVect}(k) \\
\mathrm{Num}(k)_F \arrow[r] & \mathrm{Num}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r] & \mathrm{NNum}(k)_F &
\end{tikzcd} \tag{1.11}$$
,

where Ker stands for the kernel of the respective horizontal composition towards $\mathrm{sVect}(k)$. Since by hypothesis k is of characteristic zero, the proof of [15, Thm. 1.3] shows us that the upper horizontal composition in (1.11) identifies with the functor

$$s\overline{H^*_{dR}} : \mathrm{Chow}(k)_F \longrightarrow \mathrm{sVect}(k) \quad Z \mapsto \left(\bigoplus_{n\,\mathrm{even}} H^n_{dR}(Z), \bigoplus_{n\,\mathrm{odd}} H^n_{dR}(Z) \right). \tag{1.12}$$

Hence, its kernel Ker is exactly the same as the one associated to de Rham cohomology

$$\overline{H^*_{dR}} : \mathrm{Chow}(k)_F \longrightarrow \mathrm{GrVect}(k)_{\geq 0} \qquad Z \mapsto \{H^n_{dR}(Z)\}_{n\geq 0}\,.$$

As a consequence, the pseudo-abelian envelope of $\mathrm{Chow}(k)_F/\mathrm{Ker}$ agrees with the classical category $\mathrm{Hom}(k)_F$ of homological motives; see [2, §4]. The two lower commutative squares in (1.11), combined with the fact that $\mathrm{Num}(k)_F$ and $\mathrm{NHom}(k)_F$ are idempotent complete, give then rise to the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(k)_F \arrow[r, "\Phi"] \arrow[d] & \mathrm{NHom}(k)_F \arrow[d] \\
\mathrm{Num}(k)_F \arrow[r] & \mathrm{NNum}(k)_F\,.
\end{tikzcd} \tag{1.13}$$

By construction of $\mathrm{Num}(k)_F$ and $\mathrm{NNum}(k)_F$, the kernels of the vertical functors in (1.13) are precisely the largest $\otimes$-ideals of $\mathrm{Hom}(k)_F$ and $\mathrm{NHom}(k)_F$. Hence, the commutativity of diagram (1.13) allows us to conclude that the functor Φ is

radical, *i.e.* that it preserves these largest $\otimes$-ideals. Now, note that by construction the following composition

$$\mathrm{Hom}(k)_F \xrightarrow{\Phi} \mathrm{NHom}(k)_F \xrightarrow{\overline{HP_*}} \mathrm{sVect}(k)$$

is the factorization of the above functor (1.12) through the category $\mathrm{Hom}(k)_F$ of homological motives. In particular, it is faithful. By applying [5, Prop. 2] to $\overline{HP_*}$ and $\overline{HP_*} \circ \Phi$, we obtain then an induced F-linear $\otimes$-functor $\Phi : \mathrm{Hom}^\dagger(k)^\pm_F \to \mathrm{NHom}^\dagger(k)^\pm_F$. This functor is also radical and so it gives rise to the following solid commutative square

$$\begin{tikzcd}
\mathrm{Hom}^\dagger(k)^\pm_F \arrow[rr, "\Phi"] \arrow[d] & & \mathrm{NHom}^\dagger(k)^\pm_F \arrow[d] \\
\mathrm{Num}^\dagger(k)^\pm_F \arrow[rr, "\overline{\Phi}"'] \arrow[u, dotted, bend left, "s"] & & \mathrm{NNum}^\dagger(k)^\pm_F . \arrow[u, dotted, bend right, "s^{NC}"']
\end{tikzcd}$$

The dotted arrows denote the $\otimes$-sections provided by [5, Thm. 8 a)]. Note that the unconditional motivic Galois group Gal_{dR} is given by the $\otimes$-automorphisms of the composed fiber functor

$$\omega_{dR} : \mathrm{Num}^\dagger(k)^\pm_F \xrightarrow{s} \mathrm{Hom}^\dagger(k)^\pm_F \xrightarrow{\Phi} \mathrm{NHom}^\dagger(k)^\pm_F \xrightarrow{(1.3)} \mathrm{Vect}(k) .$$

Since Φ is radical, [3, Prop. 12.2.1] and [3, Prop. 13.7.1] imply that the two $\otimes$-functors

$$\Phi \circ s : \mathrm{Num}^\dagger(k)^\pm_F \longrightarrow \mathrm{NHom}^\dagger(k)^\pm_F \qquad s^{NC} \circ \overline{\Phi} : \mathrm{Num}^\dagger(k)^\pm_F \longrightarrow \mathrm{NHom}^\dagger(k)^\pm_F$$

are naturally isomorphic (via a $\otimes$-isomorphism). As a consequence, the above fiber functor ω_{dR} becomes naturally $\otimes$-isomorphic to the following composition

$$\mathrm{Num}^\dagger(k)^\pm_F \xrightarrow{\overline{\Phi}} \mathrm{NNum}^\dagger(k)^\pm_F \xrightarrow{\omega_{HP}} \mathrm{Vect}(k) .$$

Hence, by definition of the unconditional motivic Galois groups, the functor $\overline{\Phi}$ gives rise to a well-defined group homomorphism $\mathrm{Gal}^{NC}_{HP} \to \mathrm{Gal}_{dR}$. Now, note that the Tate motive $\mathbb{Q}(1)$ clearly belongs to the category $\mathrm{Hom}(k)^\pm_F$. Hence, let us denote by $\langle \mathbb{Q}(1) \rangle$ the Tannakian subcategory of $\mathrm{Num}^\dagger(k)^\pm_F$ generated generated by $\mathbb{Q}(1)$ and write $\mathrm{Gal}_{dR}(\mathbb{Q}(1))$ for the group of $\otimes$-automorphisms of the fiber functor

$$\langle \mathbb{Q}(1) \rangle \overset{t}{\hookrightarrow} \mathrm{Num}^\dagger(k)^\pm_F \xrightarrow{\omega_{dR}} \mathrm{Vect}(k) . \tag{1.14}$$

The inclusion t gives then rise to a well-defined surjective group homomorphism $t : \mathrm{Gal}_{dR} \twoheadrightarrow \mathrm{Gal}_{dR}(\mathbb{Q}(1))$. It remains then to show that the composition

$$\mathrm{Gal}^{NC}_H \longrightarrow \mathrm{Gal}_{dR} \overset{t}{\twoheadrightarrow} \mathrm{Gal}_{dR}(\mathbb{Q}(1)) \tag{1.15}$$

is trivial. Since the categories $\mathrm{Num}^\dagger(k)^\pm_F$ and $\mathrm{NNum}^\dagger(k)^\pm_F$ are abelian semi-simple and $\overline{\Phi}$ is F-linear and additive, we conclude that $\overline{\Phi}$ is moreover *exact*, *i.e.* it preserves kernels and cokernels. Thanks to the commutative diagram (1.11) we observe that the image of $\mathbb{Q}(1)$ under $\overline{\Phi}$ is precisely the $\otimes$-unit $\underline{k}$ of $\mathrm{Num}^\dagger(k)^\pm_F$. Hence, since Φ

is also symmetric monoidal, we obtain the following commutative diagram

$$\begin{tikzcd} \langle \mathbb{Q}(1) \rangle \arrow[r, "\overline{\Phi}"] \arrow[d, hook, "t"] & \langle \underline{k} \rangle \arrow[d, hook] \\ \mathrm{Num}^\dagger(k)^\pm_F \arrow[r, "\overline{\Phi}"'] & \mathrm{NNum}^\dagger(k)^\pm_F \, , \end{tikzcd} \tag{1.16}$$

where $\langle \underline{k} \rangle$ denotes the Tannakian subcategory of $\mathrm{NNum}^\dagger(k)^\pm_F$ generated by $\underline{k}$. The group of $\otimes$-automorphisms of the fiber functor

$$\langle \underline{k} \rangle \hookrightarrow \mathrm{NNum}^\dagger(k)^\pm_F \stackrel{\omega_{HP}}{\longrightarrow} \mathrm{Vect}(k)$$

is clearly trivial, and so from the commutativity of diagram (1.16) we conclude that the above composition (1.15) is trivial. This concludes the proof of item (i).

In what concerns item (ii) we need to show that the comparison homomorphism (1.9) is surjective and that $\mathrm{Gal}_{dR}(\mathbb{Q}(1))$ agrees with the multiplicative group $\mathbb{G}_m$. Let us start with the latter claim. Note that $(\mathrm{Num}^\dagger(k)^\pm_k, w, \mathbb{Q}(1))$ is a Tate subtriple of the one described in [15, Example A.5(i)]; where w stands for the weight $\mathbb{Z}$-grading. Since by hypothesis $k = F$ this Tate triple is moreover neutral, with fiber functor given by w_{dR}. As explained in the proof of [15, Prop. 11.1], the weight $\mathbb{Z}$-grading of the Tate triple structure furnish us the following factorization

$$\begin{tikzcd} \langle \mathbb{Q}(1) \rangle \arrow[r, "\overline{(1.14)}"] \arrow[dr, "(1.14)"'] & \mathrm{GrVect}(k) \arrow[d, "U"] \\ & \mathrm{Vect}(k) \end{tikzcd}$$

of the fiber functor (1.14), where $\mathrm{GrVect}(k)$ is the category of finite dimensional $\mathbb{Z}$-graded k-vector spaces and U the forgetful functor $\{V_n\}_{n\in\mathbb{Z}} \mapsto \oplus_{n\in\mathbb{Z}} V_n$. The argument in *loc. cit.* allows us then to conclude that $\mathrm{Gal}_{dR}(\mathbb{Q}(1)) \simeq \mathbb{G}_m$. Let us now prove that the comparison homomorphism (1.9) is surjective. By applying the general [15, Prop. 11.1] to the neutral Tate-triple $(\mathrm{Num}^\dagger(k)^\pm_k, w, \mathbb{Q}(1), \omega_{dR})$ we obtain the following group isomorphism

$$\mathrm{Gal}\left((\mathrm{Num}^\dagger(k)^\pm_k/_{-\otimes\mathbb{Q}(1)})^\natural \right) \stackrel{\sim}{\longrightarrow} \mathrm{Ker}(t : \mathrm{Gal}_{dR} \twoheadrightarrow \mathbb{G}_m)\, , \tag{1.17}$$

where the left-hand-side is the group of $\otimes$-automorphisms of the composed functor

$$(\mathrm{Num}^\dagger(k)^\pm_k/_{-\otimes\mathbb{Q}(1)})^\natural \stackrel{\Psi}{\longrightarrow} \mathrm{NNum}^\dagger(k)^\pm_k \stackrel{\omega_{HP}}{\longrightarrow} \mathrm{Vect}(k)\, .$$

We now claim that Ψ is surjective. Consider the composition at the bottom of diagram (1.11), and recall that the functor $\mathrm{Num}(k)_k/_{-\otimes\mathbb{Q}(1)} \to \mathrm{NNum}(k)_k$ is fully-faithful. By first restricting ourselves to $\mathrm{Num}(k)^\pm_k$ and then by modifying the symmetry isomorphism constraints we obtain the following composition

$$\mathrm{Num}^\dagger(k)^\pm_k \longrightarrow (\mathrm{Num}(k)^\pm_k/_{-\otimes\mathbb{Q}(1)})^\dagger \stackrel{\psi}{\longrightarrow} \mathrm{NNum}^\dagger(k)^\pm_k\, . \tag{1.18}$$

The general [15, Lemma B.9] (applied to $\mathcal{C} = \mathrm{Num}(k)^\pm_k$ and $\mathcal{O} = \mathbb{Q}(1)$) furnish us a canonical $\otimes$-equivalence

$$\mathrm{Num}^\dagger(k)^\pm_k/_{-\otimes\mathbb{Q}(1)} \stackrel{\sim}{\longrightarrow} (\mathrm{Num}(k)^\pm_k/_{-\otimes\mathbb{Q}(1)})^\dagger\, .$$

Hence, since ψ is fully-faithful and $\mathrm{NNum}^\dagger(k)^{\pm}_k$ is idempotent complete we conclude that Ψ is also fully-faithful. As a consequence, we obtain an induced surjective group homomorphism

$$\mathrm{Gal}^{HP}_{dR} \twoheadrightarrow \mathrm{Gal}\left((\mathrm{Num}^\dagger(k)^{\pm}_k/_{-\otimes\mathbb{Q}(1)})^\natural\right).$$

By combining it with the isomorphism (1.17) we conclude finally that the comparison group homomorphism (1.9) is surjective. This achieves the proof of item (ii).

Let us now show item (iii). Similarly to the explanations given after Definition 1.5, whenever we assume the standard conjectures C and D the Galois group Gal_{dR} reduces to the classical motivic Galois group $\mathrm{Gal}(\mathrm{Num}^\dagger(k)_k)$. Moreover, throughout a careful verification one observes that the arguments used in the proof of item (ii) reduce to those used in the proof of [15, Thm. 1.7]. This shows item (iii) and hence concludes the proof. $\square$

2. $\otimes$-nilpotence and orbit categories

In this section, which is of general interest, we show that orbit categories are well-behaved with respect to $\otimes$-nilpotence. In what follows, $\mathcal{C}$ will be a F-linear, additive, rigid symmetric monoidal category.

Recall from [3, §7.4] that the $\otimes_{\mathrm{nil}}$-ideal of $\mathcal{C}$ is given by

$$\otimes_{\mathrm{nil}}(X,Y) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \,|\, f^{\otimes n} = 0 \text{ for } n \gg 0\}.$$

By construction, $\otimes_{\mathrm{nil}}$ is a $\otimes$-ideal. Moreover, all its ideals $\otimes_{\mathrm{nil}}(X,X) \subset \mathrm{Hom}_{\mathcal{C}}(X,X)$ are nilpotent; see [3, Lemma 7.4.2 (ii)]. As a consequence, the quotient functor $\mathcal{C} \to \mathcal{C}/\!\otimes_{\mathrm{nil}}$ is not only F-linear, additive and symmetric monoidal but moreover conservative; see [8, Lemma 3.1]. Furthermore, since idempotents can be lifted along nilpotent ideals (see [8, Prop. 3.2]), whenever $\mathcal{C}$ is idempotent complete so it is the quotient category $\mathcal{C}/\!\otimes_{\mathrm{nil}}$.

Now, let $\mathcal{O}$ be a $\otimes$-invertible object of $\mathcal{C}$. Recall from [17, §7] the construction of the orbit category $\mathcal{C}/_{-\otimes\mathcal{O}}$. It has the same objects as $\mathcal{C}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(X,Y) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y\otimes\mathcal{O}^{\otimes j}).$$

The composition law is induced by $\mathcal{C}$. By construction, $\mathcal{C}/_{-\otimes\mathcal{O}}$ is F-linear, additive, and it comes equipped with a canonical projection functor $\tau: \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$. Moreover, τ is endowed with a natural 2-isomorphism $\tau\circ(-\otimes\mathcal{O}) \overset{\sim}{\Rightarrow} \tau$ and is 2-universal among all such functors. As proved in [17, Lemma 7.3], the orbit category $\mathcal{C}/_{-\otimes\mathcal{O}}$ inherits a natural symmetric monoidal structure from $\mathcal{C}$ making the projection functor τ symmetric monoidal. On objects this monoidal structure is the same as the one on $\mathcal{C}$. On morphisms it is defined as the unique bilinear map

$$\bigoplus_{j\in\mathbb{Z}}\mathrm{Hom}_{\mathcal{C}}(X,Y\otimes\mathcal{O}^{\otimes j})\times\bigoplus_{j\in\mathbb{Z}}\mathrm{Hom}_{\mathcal{C}}(Z,W\otimes\mathcal{O}^{\otimes j}) \longrightarrow \bigoplus_{j\in\mathbb{Z}}\mathrm{Hom}_{\mathcal{C}}(X\otimes Z,(Y\otimes W)\otimes\mathcal{O}^{\otimes j})$$

sending the homogeneous maps $f_i: X \to Y\otimes\mathcal{O}^{\otimes i}$ and $g_s: Z \to W\otimes\mathcal{O}^{\otimes s}$ to the homogeneous map

$$(f\otimes g)_{(i+s)}: X\otimes Z \stackrel{f_i\otimes g_s}{\longrightarrow} Y\otimes\mathcal{O}^{\otimes i}\otimes W\otimes\mathcal{O}^{\otimes s} \stackrel{\sim}{\longrightarrow} (Y\otimes W)\otimes\mathcal{O}^{\otimes(i+s)},$$

where the right-hand-side map is the commutativity isomorphism constraint.

Proposition 2.1. *The identity functor on $\mathcal{C}$ gives rise to a $\otimes$-equivalence*

$$\Theta : (\mathcal{C}/\otimes_{\mathrm{nil}})/_{-\otimes\mathcal{O}} \xrightarrow{\sim} (\mathcal{C}/_{-\otimes\mathcal{O}})/\otimes_{\mathrm{nil}}$$

making the following diagram commutative

$$\begin{tikzcd}
\mathcal{C} \arrow[rr] \arrow[dd] & & \mathcal{C}/_{-\otimes\mathcal{O}} \arrow[dl] \arrow[dd] \\
& (\mathcal{C}/\otimes_{\mathrm{nil}})/_{-\otimes\mathcal{O}} \arrow[dr, "\Theta", "\sim"'] & \\
\mathcal{C}/\otimes_{\mathrm{nil}} \arrow[ur] \arrow[rr] & & (\mathcal{C}/_{-\otimes\mathcal{O}})/\otimes_{\mathrm{nil}}
\end{tikzcd} \tag{2.2}$$

Proof. Let us start by showing that Θ is well-defined. On one hand, an element of $\mathrm{Hom}_{(\mathcal{C}/\otimes_{\mathrm{nil}})/_{-\otimes\mathcal{O}}}(X,Y)$ consists of a sequence $\{[f_j]\}_{j\in\mathbb{Z}}$, where $f_j : X \to Y \otimes \mathcal{O}^{\otimes j}$ is a morphism in $\mathcal{C}$, $[f_j]$ is the class of f_j in $\mathcal{C}/\otimes_{\mathrm{nil}}$ and $[f_j] = 0$ for $|j| \gg 0$. On the other hand, an element of $\mathrm{Hom}_{(\mathcal{C}/_{-\otimes\mathcal{O}})/\otimes_{\mathrm{nil}}}(X,Y)$ is the class $[\{f_j\}_{j\in\mathbb{Z}}]$ of a sequence $\{f_j\}_{j\in\mathbb{Z}}$, where $f_j : X \to Y \otimes \mathcal{O}^{\otimes j}$ is a morphism in $\mathcal{C}$ and $f_j = 0$ for $|j| \gg 0$. Hence, in order to prove that the assignment

$$\{[f_j]\}_{j\in\mathbb{Z}} \mapsto [\{g_j\}_{j\in\mathbb{Z}}] \quad \text{where} \quad g_j := \begin{cases} f_j & \text{if} \quad [f_j] \neq 0 \\ 0 & \text{if} \quad [f_j] = 0 \end{cases}$$

gives rise to a well-defined functor it suffices to show that for every $i \in \mathbb{Z}$ the composition

$$\mathrm{Hom}_{\mathcal{C}}(X, Y \otimes \mathcal{O}^{\otimes i}) \overset{\iota}{\hookrightarrow} \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y \otimes \mathcal{O}^{\otimes j}) \to \mathrm{Hom}_{(\mathcal{C}/_{-\otimes\mathcal{C}})/\otimes_{\mathrm{nil}}}(X,Y)$$

becomes trivial when restricted to $\otimes_{\mathrm{nil}}(X, Y \otimes \mathcal{O}^{\otimes i}) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y \otimes \mathcal{O}^{\otimes i})$. Let $f : X \to Y \otimes \mathcal{O}^{\otimes i}$ be a morphism in $\otimes_{\mathrm{nil}}(X, Y \otimes \mathcal{O}^{\otimes i})$. By definition, there exist a natural number $n \gg 0$ such that $f^{\otimes n} = 0$. Hence, by definition of the symmetric monoidal structure on $\mathcal{C}/_{-\otimes\mathcal{O}}$, we observe that $\iota(f)^{\otimes n}$ agrees with the image of $f^{\otimes n}$ under the inclusion

$$\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, Y^{\otimes n} \otimes \mathcal{O}^{\otimes(n\cdot i)}) \hookrightarrow \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, Y^{\otimes n} \otimes \mathcal{O}^{\otimes j})\,.$$

Since $f^{\otimes n} = 0$, the morphism $\iota(f)$ is $\otimes$-nilpotent and so it vanishes in the Hom-set $\mathrm{Hom}_{(\mathcal{C}/_{-\otimes\mathcal{O}})/\otimes_{\mathrm{nil}}}(X,Y)$. As a consequence, the above functor Θ is well-defined. By construction it is full and symmetric monoidal. Since the categories $(\mathcal{C}/\otimes_{\mathrm{nil}})/_{-\otimes\mathcal{O}}$ and $(\mathcal{C}/_{-\otimes\mathcal{O}})/\otimes_{\mathrm{nil}}$ have exactly the same objects we observe that Θ is moreover (essentially) surjective. It remains then to show that it is faithful. In order to prove this, consider an element $\{[f_j]\}_{j\in\mathbb{Z}}$ of $\mathrm{Hom}_{(\mathcal{C}/\otimes_{\mathrm{nil}})/_{-\otimes\mathcal{O}}}(X,Y)$. Let max and min be the uniquely determined integers such that

$$[f_j] = \begin{cases} 0 & \text{when} \quad j > \mathrm{max} \\ 0 & \text{when} \quad j < \mathrm{min} \\ \neq 0 & \text{when} \quad j = \mathrm{min}, \mathrm{max}\,. \end{cases}$$

Suppose that $\Theta(\{f_j]\}_{j\in\mathbb{Z}}) = 0$. By definition, there exists then a natural number $m \gg 0$ such that $(\{g_j\}_{j\in\mathbb{Z}})^{\otimes m} = 0$ in $\mathcal{C}/_{-\otimes\mathcal{O}}$. The symmetric monoidal structure on $\mathcal{C}/_{-\otimes\mathcal{O}}$ allows us then to conclude that $g_{\mathrm{min}}^{\otimes m} = f_{\mathrm{min}}^{\otimes m} = 0$ and $g_{\mathrm{max}}^{\otimes m} = f_{\mathrm{max}}^{\otimes m}$ in $\mathcal{C}$. As a consequence, we obtain the following equality $[f_{\mathrm{min}}] = [f_{\mathrm{max}}] = 0$. Now, by applying the above argument recursively, we conclude that $[f_j] = 0$ for all $j \in \mathbb{Z}$. This shows that the functor Θ is faithful and so the proof is finished. $\square$

3. Smash-nilpotence in the noncommutative world

Throughout this section we will assume that F is a field of characteristic zero. By construction, the categories $\mathrm{Chow}(k)_F$ and $\mathrm{NChow}(k)_F$ satisfy all the conditions of §2, *i.e.* they are F-linear, additive, and rigid symmetric monoidal. Let us denote by $\mathrm{Voev}(k)_F$ and $\mathrm{NVoev}(k)_F$ the associated quotient categories $\mathrm{Chow}(k)_F/\otimes_{\mathrm{nil}}$ and $\mathrm{NChow}(k)_F/\otimes_{\mathrm{nil}}$. Note that since $\mathrm{Chow}(k)_F$ and $\mathrm{NChow}(k)_F$ are idempotent complete so are the quotient categories $\mathrm{Voev}(k)_F$ and $\mathrm{NVoev}(k)_F$. The category $\mathrm{Voev}(k)_F$ agrees with the one introduced originally by Voevodsky [18] using smash-nilpotence of algebraic cycles. The category $\mathrm{NVoev}(k)_F$ should be considered as its noncommutative analogue. This is justified by the following result.

Proposition 3.1. *There exist F-linear, additive, $\otimes$-functors $R, R_{\otimes_{\mathrm{nil}}}, R_{\mathcal{N}}$ making the following diagram commutative*

$$
\begin{tikzcd}
\mathrm{Chow}(k)_F \arrow[r, "\tau"] \arrow[d] & \mathrm{Chow}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r, "R"] \arrow[d] & \mathrm{NChow}(k)_F \arrow[d] \\
\mathrm{Voev}(k)_F \arrow[r, "\tau"] \arrow[d] & \mathrm{Voev}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r, "R_{\otimes_{\mathrm{nil}}}"] \arrow[d] & \mathrm{NVoev}(k)_F \arrow[d] \\
\mathrm{Num}(k)_F \arrow[r, "\tau"'] & \mathrm{Num}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r, "R_{\mathcal{N}}"'] & \mathrm{NNum}(k)_F\,.
\end{tikzcd}
\tag{3.2}
$$

Moreover, $R, R_{\otimes_{\mathrm{nil}}}$ and $R_{\mathcal{N}}$ are fully-faithful.

Intuitively speaking, Proposition 3.1 formalizes the conceptual idea that all the categories of pure motives can be embedded into their noncommutative analogues after factoring out by the action of the Tate motive.

Proof. Recall from [15, Prop. 4.4] the construction of the following commutative diagram

$$
\begin{tikzcd}
\mathrm{Chow}(k)_F \arrow[r, "\tau"] \arrow[d] & \mathrm{Chow}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r, "R"] \arrow[d] & \mathrm{NChow}(k)_F \arrow[d] \\
\mathrm{Num}(k)_F \arrow[r, "\tau"'] & \mathrm{Num}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r, "R_{\mathcal{N}}"'] & \mathrm{NNum}(k)_F\,,
\end{tikzcd}
$$

where R and $R_{\mathcal{N}}$ are fully-faithful functors. The categories $\mathrm{Num}(k)_F$ and $\mathrm{NNum}(k)_F$ are obtained, respectively, from $\mathrm{Chow}(k)_F$ and $\mathrm{NChow}(k)_F$ in two steps: first we factor out by the largest $\otimes$-ideal (distinct from the entire category) and then we pass to the pseudo-abelian envelope. Hence, we obtain the following induced commutative diagram

$$
\begin{tikzcd}
\mathrm{Chow}(k)_F \arrow[r, "\tau"] \arrow[d] & \mathrm{Chow}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r, "R"] \arrow[d] & \mathrm{NChow}(k)_F \arrow[d] \\
\mathrm{Voev}(k)_F \arrow[r] \arrow[d] & (\mathrm{Chow}(k)_{F/_{-\otimes\mathbb{Q}(1)}})/\otimes_{\mathrm{nil}} \arrow[r, "\overline{R_{\otimes_{\mathrm{nil}}}}"] \arrow[d] & \mathrm{NVoev}(k)_F \arrow[d] \\
\mathrm{Num}(k)_F \arrow[r, "\tau"'] & \mathrm{Num}(k)_{F/_{-\otimes\mathbb{Q}(1)}} \arrow[r, "R_{\mathcal{N}}"'] & \mathrm{NNum}(k)_F\,.
\end{tikzcd}
\tag{3.3}
$$

Since the functor R is fully-faithful, we conclude that so it is the induced functor $\overline{R_{\otimes_{\mathrm{nil}}}}$. By pre-composing $\overline{R_{\otimes_{\mathrm{nil}}}}$ with the canonical functor Θ of diagram (2.2) (applied to the upper left square of (3.3) with $\mathcal{C} = \mathrm{Chow}(k)_F$ and $\mathcal{O} = \mathbb{Q}(1)$) we obtain then the functor $R_{\otimes_{\mathrm{nil}}}$ and the above commutative diagram (3.2). □

4. Noncommutative nilpotence conjecture

Throughout this section we will assume that F is a field of characteristic zero. Given a smooth projective k-scheme Z, recall from [2, §3.2] the definition of the F-vector spaces $\mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(Z)_F$ and $\mathcal{Z}^*_{\mathrm{num}}(Z)_F$ of algebraic cycles.

Voevodsky's nilpotence conjecture $V(Z)$ **(see** [18, Conj. 4.2]**):** We have the following equality $\mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(Z)_F = \mathcal{Z}^*_{\mathrm{num}}(Z)_F$.

Let $\mathcal{A}$ be a smooth and proper dg category $\mathcal{A}$. Recall that by construction of the category of noncommutative Chow motives we have the following identification

$$K_0(\mathcal{A})_F \simeq \mathrm{Hom}_{\mathrm{NChow}(k)_F}(\underline{k}, \mathcal{A}) .$$

The $\otimes$-ideal $\otimes_{\mathrm{nil}}$ gives then rise to a well-defined equivalence relation on $K_0(\mathcal{A})_F$ that we will denote by $\sim_{\otimes_{\mathrm{nil}}}$. Recall from [15, §10] that $K_0(\mathcal{A})_F$ is also endowed with a numerical equivalence relation denoted by $\sim_{\mathrm{num}}$. Motivated by Voevodsky's nilpotence conjecture we introduce the following

Noncommutative nilpotence conjecture $V_{NC}(\mathcal{A})$**:** We have the following equality $K_0(\mathcal{A})_F/\!\sim_{\otimes_{\mathrm{nil}}} = K_0(\mathcal{A})_F/\!\sim_{\mathrm{num}}$.

Now, recall from [1, §3] that in any category which is F-linear (with F of characteristic zero), idempotent complete, and symmetric monoidal, we have the well-defined notions of *even*, *odd*, and *finite dimensional* object. This applies in particular to the categories $\mathrm{Chow}(k)_F$ and $\mathrm{NChow}(k)_F$.

Kimura-O'Sullivan conjecture $KS(Z)$ **(see** [2, Conj. 12.1.2.1]**):** The Chow motive Z is finite dimensional.

Noncommutative finiteness conjecture $KS_{NC}(\mathcal{A})$**:** The noncommutative Chow motive $\mathcal{A}$ is finite dimensional.

The web of relations between all the above conjectures is the following:

Theorem 4.1. (i) *Conjecture $V(Z)$ is equivalent to conjecture $V_{NC}(\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z))$.*
(ii) *Conjecture $KS(Z)$ implies conjecture $KS_{NC}(\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z))$.*
(iii) *Assume that k is a field extension of F or vice-versa. Then, conjectures $V_{NC}((\mathcal{A}^{\mathsf{op}})^{\otimes n} \otimes \mathcal{A}^{\otimes n}), n \geq 1$, combined with the noncommutative standard conjecture $C_{NC}(\mathcal{A})$, imply conjecture $KS_{NC}(\mathcal{A})$.*
(iv) *If conjecture KS_{NC} holds for every smooth and proper dg category and all the $\otimes$-functors $\mathrm{NChow}(k)_F \to \mathrm{sVect}(L)$ (with L a field extension of F) factor through $\mathrm{NNum}(k)_F$, then conjecture V_{NC} holds also for every smooth and proper dg category.*

Remark 4.2. Item (i) shows us that when restricted to the commutative world the noncommutative nilpotence conjecture reduces to Voevodsky's original conjecture. This not only justifies its correctness but moreover stimulates the use of noncommutative ideas and methods in order to prove Voevodsky's original conjecture. In what concerns item (ii) it is unclear to the authors if the converse statement holds. Roughly speaking, this would amount to "lift" a decomposition of $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z)$ into a

decomposition of Z. Item (iii) shows us that the finite dimensionality of a noncommutative Chow motive can be recovered from the algebraicity of the Künneth projectors and from the agreement between $\otimes$-nilpotence and numerical equivalence. Finally, item (iv) should be considered as a global converse statement of item (iii), with C_{NC} replaced by a strong variant of the noncommutative standard conjecture D_{NC}.

Proof. Let us start by proving item (i). Recall from [17, Thm. 1.1] that the image of Z under the composed functor

$$\mathrm{Chow}(k)_F \xrightarrow{\tau} \mathrm{Chow}(k)_{F}/_{-\otimes\mathbb{Q}(1)} \xrightarrow{R} \mathrm{NChow}(k)_F \tag{4.3}$$

identifies naturally with the noncommutative Chow motive $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z)$. Similarly, the image of $\mathrm{Spec}(k)$ identifies with $\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(\mathrm{Spec}(k))$ which is Morita equivalent to $\underline{k}$. Hence, the lower right square of (3.2) gives rise to the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Voev}(k)_F/_{-\otimes\mathbb{Q}(1)}}(\mathrm{Spec}(k), Z) & \longrightarrow & \mathrm{Hom}_{\mathrm{NVoev}(k)_F}(\underline{k}, \mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Num}(k)_F/_{-\otimes\mathbb{Q}(1)}}(\mathrm{Spec}(k), Z) & \longrightarrow & \mathrm{Hom}_{\mathrm{NNum}(k)_F}(\underline{k}, \mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z)) \,. \end{array}$$

By construction of the categories $\mathrm{Voev}(k)_F$, $\mathrm{Num}(k)_F$, $\mathrm{NVoev}(k)_F$, and $\mathrm{NNum}(k)_F$ we observe that the preceding commutative diagram reduces to

$$\begin{array}{ccc} \mathcal{Z}^*_{\otimes_{\mathrm{nil}}}(Z)_F & \xrightarrow{\sim} & K_0(\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z))_F/_{\sim_{\otimes_{\mathrm{nil}}}} \\ \downarrow & & \downarrow \\ \mathcal{Z}^*_{\mathrm{num}}(Z)_F & \xrightarrow[\sim]{} & K_0(\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z))_F/_{\sim_{\mathrm{num}}} \,, \end{array} \tag{4.4}$$

where the horizontal maps are isomorphisms since $R_{\otimes_{\mathrm{nil}}}$ and $R_{\mathcal{N}}$ are fully-faithful. Now, let us assume Voevodsky's nilpotence conjecture $V(Z)$, *i.e.* let us assume that the left vertical map in diagram (4.4) is an isomorphism. The commutativity of this diagram then implies that the right vertical map is also an isomorphism. This is precisely the statement of the noncommutative nilpotence conjecture $V_{NC}(\mathcal{D}^{\mathsf{dg}}_{\mathsf{perf}}(Z))$ and so the proof is finished.

Item (ii) follows from the fact that the functor (4.3) is F-linear, additive, and symmetric monoidal and that finite dimensionality is preserved under F-linear additive $\otimes$-functors.

Let us now prove item (iii). Note first that the category $\mathrm{NChow}(k)_F$ is endowed with three distinct $\otimes$-ideals: the ideal $\otimes_{\mathrm{nil}}$ described in §2, the kernel $\mathrm{Ker}(\overline{HP_*})$ of the $\otimes$-functor (1.1), and the largest $\otimes$-ideal $\mathcal{N}$ of those noncommutative correspondences which are numerically equivalent to zero. Since there are no non-trivial $\otimes$-nilpotent morphisms in $\mathrm{sVect}(k)$ the following inclusions hold $\otimes_{\mathrm{nil}} \subseteq \mathrm{Ker}(\overline{HP_*}) \subseteq \mathcal{N}$. As a consequence, the F-algebra $\mathrm{Hom}_{\mathrm{NChow}(k)_F}(\mathcal{A},\mathcal{A})$ inherits these distinct ideals

$$\otimes_{\mathrm{nil}}(\mathcal{A},\mathcal{A}) \subseteq \mathrm{Ker}(\overline{HP_*})(\mathcal{A},\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A},\mathcal{A}) \,. \tag{4.5}$$

Now, since the dual of a smooth and proper dg category $\mathcal{A}$ is its opposite dg category $\mathcal{A}^{\mathsf{op}}$ (see [6, Thm. 4.8]), we have the following identification

$$\mathrm{Hom}_{\mathrm{NChow}(k)_F}(\mathcal{A},\mathcal{A}) \simeq \mathrm{Hom}_{\mathrm{NChow}(k)_F}(\underline{k}, \mathcal{A}^{\mathsf{op}}\otimes\mathcal{A})\,.$$

Assuming conjecture $V_{NC}(\mathcal{A}^{\mathsf{op}}\otimes\mathcal{A})$ we then conclude that the above three ideals (4.5) are in fact the same. Hence, we obtain the following equalities

$$\mathrm{End}_{\mathrm{NChow}(k)_F}(\mathcal{A}) = \mathrm{End}_{\mathrm{NHom}(k)_F}(\mathcal{A}) = \mathrm{End}_{\mathrm{NNum}(k)_F}(\mathcal{A})\,. \tag{4.6}$$

Moreover, assuming the noncommutative standard conjecture $C_{NC}(\mathcal{A})$ we have an idempotent $\underline{\pi}^+_{\mathcal{A}} \in \mathrm{End}_{\mathrm{NHom}(k)_F}(\mathcal{A})$ which is mapped to the Künneth projector

$$\pi^+_{\mathcal{A}}: \overline{HP_*} \twoheadrightarrow \overline{HP_*}^+(\mathcal{A}) \hookrightarrow \overline{HP_*}(\mathcal{A})\,.$$

As explained in §2, the ideal $\otimes_{\mathrm{nil}}(\mathcal{A},\mathcal{A})$ is nilpotent. Hence, the idempotent $\underline{\pi}^+_{\mathcal{A}}$ can be lifted along the projection

$$\mathrm{End}_{\mathrm{NChow}(k)_F}(\mathcal{A}) \twoheadrightarrow \mathrm{End}_{\mathrm{NHom}(k)_F}(\mathcal{A})\,.$$

The category $\mathrm{NChow}(k)_F$ is idempotent complete and so let us denote by $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ the associated direct sum decomposition. It remains then to show that $\mathcal{A}_+$ (resp. $\mathcal{A}_-$) is an even (resp. odd) object. The arguments are similar in both cases and so we restrict ourselves to $\mathcal{A}_+$. All the functors in the following composition are symmetric monoidal

$$\mathrm{NChow}(k)_F \longrightarrow \mathrm{NVoev}(k)_F \longrightarrow \mathrm{NHom}(k)_F \xrightarrow{\overline{HP_*}} \mathrm{sVect}(K)\,.$$

Since $\overline{HP_*}$ is moreover faithful and every object in $\mathrm{sVect}(k)$ is even, we conclude that there exists an integer $m \geq 0$ such that the exterior power $\wedge^m(\mathcal{A}_+)$ vanishes in $\mathrm{NHom}(k)_F$. Now, assuming the conjectures $V_{NC}((\mathcal{A}^{\mathsf{op}})^{\otimes n}\otimes\mathcal{A}^{\otimes n}), n\geq 1$, we conclude (using the same argument as above and the fact that $(\mathcal{A}^{\mathsf{op}})^{\otimes n} = (\mathcal{A}^{\otimes n})^{\mathsf{op}}$) that the equalities (4.6) hold also when $\mathcal{A}$ is replaced by $\mathcal{A}^{\otimes n}, n \geq 1$. As a consequence, the functor $\mathrm{NVoev}(k)_F \to \mathrm{NHom}(k)_F$ becomes fully-faithful when restricted to the direct summands of the noncommutative motives $\mathcal{A}^{\otimes n}, n \geq 1$. Since by definition $\wedge^m(\mathcal{A}_+)$ is a direct summand of $\mathcal{A}^{\otimes m}$ we conclude that the exterior power $\wedge^m(\mathcal{A}_+)$ already vanishes in $\mathrm{NVoev}(k)_F$. Finally, the conservativity of the projection functor $\mathrm{NChow}(k)_F \to \mathrm{NVoev}(k)_F$ (see §2) allows us to conclude that $\mathcal{A}_+$ is even in $\mathrm{NChow}(k)_F$. This concludes the proof of item (iii).

Let us now prove item (iv). Recall that by construction every noncommutative Chow motive is a direct factor of a smooth and proper dg category. Since finite dimensionality is stable under direct factors, we then conclude from our hypothesis that every noncommutative Chow motive is finite dimensional. Hence, $\mathrm{NChow}(k)_F$ is a Kimura-O'Sullivan category in the sense of [1, Def. 3.3]. As explained in [1, §3.7], the $\otimes$-ideal $\otimes_{\mathrm{nil}}$ can then be expressed as the intersection of the kernels of all the $\otimes$-functors $\mathrm{NChow}(k)_F \to \mathrm{sVect}(L)$ (with L a field extension of F). Since by hypothesis all these functors factor through $\mathrm{NNum}(k)_F$ we conclude that $\mathcal{N} \subset \otimes_{\mathrm{nil}}$. Now, recall from [15, §3.2] that $\mathcal{N}$ is the largest $\otimes$-ideal of $\mathrm{NChow}(k)_F$ (distinct from the entire category). Hence, $\mathcal{N} = \otimes_{\mathrm{nil}}$ and so we conclude that the noncommutative nilpotent conjecture V_{NC} holds for every smooth and proper dg category. This concludes the proof of item (iv) and hence of the theorem. □

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References

[1] Y. André, *Motifs de dimension finie (d'après S.-I. Kimura, P. O'Sullivan. . .)*. Astérisque **299** (2005). Exp. No. 929, viii, 115–145, Séminaire Bourbaki. Vol. 2003/2004.

[2] ———, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*. Panoramas et Synthèses **17**. Société Mathématique de France, Paris, 2004.

[3] Y. André and B. Kahn, *Nilpotence, radicaux et structures monoïdales.* (French) Rend. Sem. Mat. Univ. Padova **108** (2002), 107–291.

[4] ———, *Erratum: Nilpotence, radicaux et structures monoïdales.* (French) Rend. Sem. Mat. Univ. Padova **108** (2002), 125-128.

[5] ———, *Construction inconditionnelle de groupes de Galois motiviques.* (French) C. R. Math. Acad. Sci. Paris **334** (2002), no. 11, 989–994.

[6] D.-C. Cisinski and G. Tabuada, *Symmetric monoidal structure on Non-commutative motives.* Available at arXiv:1001.0228v2. To appear in Journal of K-theory.

[7] P. Deligne, *Catégories tannakiennes.* The Grothendieck Festschift, vol **II**, 111–195, Boston: Birkhäuser.

[8] F. Ivorra, *Finite dimensional motives and applications following Kimura, O'Sullivan and others.* Notes available at http://www.uni-due.de/hm0035/SurveyIHES.

[9] B. Keller, *On differential graded categories.* International Congress of Mathematicians (Madrid), Vol. II, 151–190, Eur. Math. Soc., Zürich, 2006.

[10] S. Kleiman, *Algebraic cycles and Weil conjectures.* Dix exposés sur la cohomologie des schémas. North-Holland, Amsterdam; Masson, Paris (1968), 359–386.

[11] M. Kontsevich, *Noncommutative motives.* Talk at the Institute for Advanced Study on the occasion of the 61^{st} birthday of Pierre Deligne, October 2005. Video available at http://video.ias.edu/Geometry-and-Arithmetic.

[12] V. Lunts and D. Orlov, *Uniqueness of enhancement for triangulated categories.* J. Amer. Math. Soc. **23** (2010), 853–908.

[13] M. Marcolli and G. Tabuada, *Noncommutative motives, numerical equivalence, and semi-simplicity.* Available at arXiv:1105.2950.

[14] ———, *Kontsevich's category of noncommutative numerical motives.* Available at arXiv:1108.3785.

[15] ———, *Noncommutative numerical motives, Tannakian structures, and motivic Galois groups.* Available at arXiv:1110.2438.

[16] P. O'Sullivan, *The structure of certain rigid tensor categories.* C. R. Math. Acad. Sci. Paris **340** (2005), no. 8, 557–562.

[17] G. Tabuada, *Chow motives versus noncommutative motives.* Available at arXiv:1103.0200. To appear in Journal of Noncommutative Geometry.

[18] V. Voevodsky, *A nilpotence theorem for cycles algebraically equivalent to zero.* Int. Math. Res. Notices (1995), no. 4, 187–198.

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